

On regularity properties of Bessel flow

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Abstract

We study the differentiability of Bessel flow $\rho : x \rightarrow \rho_t^x$, where $(\rho_t^x)_{t \geq 0}$ is BES $^x(\delta)$ process of dimension $\delta > 1$ starting from x . For $\delta \geq 2$ we prove the existence of bicontinuous derivatives in P-a.s. sense at $x \geq 0$ and we study the asymptotic behaviour of the derivatives at $x = 0$. For $1 < \delta < 2$ we prove the existence of a modification of Bessel flow having derivatives in probability sense at $x \geq 0$. We study the asymptotic behaviour of the derivatives at $t = \tau_0(x)$ where $\tau_0(x)$ is the first zero of $(\rho_t^x)_{t \geq 0}$.

1 Introduction

The regularity of flows of diffusion processes is an important problem related to the stability of solutions of SDEs with respect to the initial value. This problem is well-studied when the coefficients of the diffusion equation are regular(cf.KUNITA [7], PROTTER[9]). Some results for the non-Lipschitz case is given in REN, ZHANG [11].

As is well-known, Bessel squared process of dimension $\delta > 0$, denoted by BESQ $^{x^2}(\delta)$, starting from x^2 , is the unique strong solution of the following stochastic differential equation: for all $x \geq 0$ and $t > 0$

$$X_t^x = x^2 + 2 \int_0^t \sqrt{X_s^x} d\beta_s + \delta t, \quad (1)$$

where $\beta = (\beta_t)_{t \geq 0}$ is standard Brownian motion. For $x > \gamma > 0$ with fixed γ , $X^x = (X_t^x)_{t \geq 0}$ is diffusion process with locally Lipschitz coefficients on $(0, +\infty)$. Moreover, the derivatives of diffusion coefficients with respect to initial value are also locally Lipschitz on the same set. It gives, using the comparison theorem, that the flow of BESQ $^{x^2}(\delta)$ processes with $x > \gamma$, is a diffeomorphisme up to explosion time for derivatives, which is

$$\tau_0(\gamma) = \inf\{t \geq 0 : X_t^\gamma = 0\}$$

where $\inf\{\emptyset\} = \infty$. We remark that in the case $\delta \geq 2$ we have $P(\tau_0(\gamma) = \infty) = 1$, and in the case $1 < \delta < 2$ we get $P(\tau_0(\gamma) < \infty) = 1$. In general, we cannot expect to establish some regularity properties after explosion time for the derivatives. But BESQ $^{x^2}(\delta)$ process is a very special case in which this study may be possible. It should be noticed that being particular, BESQ processes appear relatively often: it is so for radial part squared of Brownian motion;

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the laws of some local times for Brownian motion are related to BESQ process, and the same is true for some processes related with running maximum of Brownian motion (see BORODIN, SALMINEN [2]). Another important example is given by the trace of Wishart process which is also BESQ process (see BRU[1]). In this context we should also mention Dunkl process which radial part squared is also BESQ process (see L. GALLARDO, M. YOR[4],[5]).

The same comments can be made for $BES^x(\delta)$ with $\delta > 0$. This process is related to $BESQ^{x^2}(\delta)$ in the following way: for $t \geq 0$

$$\rho_t^x = \sqrt{X_t^x}.$$

It is well known (see, for instance [10], chapter XI) that for $\delta > 1$, the $BES^x(\delta)$ process is the solution of the following differential equation: for all $x \geq 0$ and $t > 0$

$$\rho_t^x = x + \beta_t + \frac{(\delta - 1)}{2} \int_0^t \frac{1}{\rho_s^x} ds \quad (2)$$

where $\beta = (\beta_t)_{t \geq 0}$ is standard Brownian motion. For $\delta = 1$ the $BES^x(\delta)$ process satisfies: for all $x \geq 0$ and $t > 0$

$$\rho_t^x = x + \beta_t + L_t^0(x) \quad (3)$$

where $L_t^0(x)$ is local time of $BES^x(\delta)$ process at zero. For $0 < \delta < 1$ $BES^x(\delta)$ process verify: for all $x \geq 0$ and $t > 0$

$$\rho_t^x = x + \beta_t + \frac{(\delta - 1)}{2} \text{v.p.} \int_0^t \frac{1}{\rho_s^x} ds \quad (4)$$

where the integral in the right-hand side is understanding is in v.p. sense.

The structure of $BES^x(\delta)$ process is simpler then the one of $BESQ^{x^2}(\delta)$ process in a sense that the equations (2), (3), (4) do not contain a stochastic integral. This is the reason why we focuss our study on the flow of $BES^x(\delta)$ processes. We remark that some indications related to the regularity property of Bessel flow with $\delta \geq 2$ and $x > 0$ can be found in HIRSCH, SONG[6].

The aim of this paper is to study the regularity property of the flow of $BES^x(\delta)$ processes with $\delta > 1$. We will distinguish the cases of $\delta \geq 2$ and $1 < \delta < 2$, and inside of them also the cases $x > 0$ and $x \geq 0$.

Theorem 1.1. *For $\delta \geq 2$, the flow of $BES^x(\delta)$ processes has (P-a.s.) derivatives of all orders with respect to x for $x > 0$ which are bicontinuous in (x, t) on the set $]0, +\infty[\times]0, +\infty[$.*

To prove Theorem 1.1 we reduce first the problem to the case of $x > \gamma > 0$, then we do localisation and we use the classical results. The case $x \geq 0$ is very different from the case $x > 0$ from point of view of properties and, then also from technical point of view. In the case $x \geq 0$ the mentionned above procedure does not work and we have to use some identity in law and some fine asymptotics of Spitzer type to conclude (cf. SPITZER[13], MESSULAM P., YOR M.[8]).

Theorem 1.2. *For $\delta = 2$, the flow of $BES^x(\delta)$ processes has (P-a.s.) derivatives of all orders n at $x = 0$ (and a fortiori for $x > 0$). These derivatives are bicontinuous in (x, t) on the set $[0, +\infty[\times]0, +\infty[$. Moreover, for the derivatives of Bessel flow $(\rho_t^x)_{t \geq 0, x \geq 0}$ the following asymptotic relations hold.*

a) For $n \geq 1$ uniformly on compacts in t of $(0, +\infty)$ and P -a.s.

$$\lim_{x \rightarrow 0+} \left(\ln \left(\frac{\partial^n \rho_t^x}{\partial x^n} \right) / \ln x \right) = +\infty.$$

b) For $n \geq 1$ the convergence in law sense holds:

$$\lim_{x \rightarrow 0+} \left(\ln \left(\frac{\partial^n \rho_t^x}{\partial x^n} \right) / (\ln x)^2 \right) = -T_1(\beta)/2$$

where $T_1(\beta)$ is the first passage time of the level 1 for standard Brownian motion,

c) Uniformly on compacts in t of $(0, +\infty)$ and P -a.s.

$$\lim_{x \rightarrow 0+} \left(x^{n-1} \frac{\partial^n \rho_t^x}{\partial x^n} / \frac{\partial \rho_t^x}{\partial x} \right) = U_{n-1}$$

where $U_{n-1} = U_1(U_1 - 1) \cdots (U_1 - n + 2)$ and U_1 is a random variable given by (24) with $\nu = 1$.

d) For $T > \epsilon > 0$ and $0 < \gamma < 1/(n-1)$ with the same U_{n-1} as in c)

$$\lim_{x \rightarrow 0+} E \left(\sup_{\epsilon \leq t \leq T} x^{n-1} \left| \frac{\partial^n \rho_t^x}{\partial x^n} / \frac{\partial \rho_t^x}{\partial x} \right| \right)^\gamma = E(|U_{n-1}|^\gamma)$$

For $\delta > 2$, the flow of $BES^x(\delta)$ processes has (P -a.s.) at $x = 0$ the derivatives only up to the order $n < n(\delta)$ where $n(\delta) = 2 + \frac{1}{\delta - 2}$ (and of all orders for $x > 0$). These derivatives are bicontinuous in (x, t) on the set $[0, +\infty[\times]0, +\infty[$. Moreover for $n < n(\delta)$ we have:

a') uniformly on compacts in t of $(0, +\infty)$ and P -a.s.

$$\lim_{x \rightarrow 0+} \left(\ln \left(\frac{\partial^n \rho_t^x}{\partial x^n} \right) / \ln x \right) = n(\delta) - n,$$

and also the property c) and the property d) with $0 < \gamma < \nu/(n-1)$ and $\nu = 2\delta - 3$ in (24).

Remark 1. For the regularity at $x = 0$ we have the following picture. If $\delta \geq 3$ then the flow has only two derivatives in P -a.s. sense and no derivatives of order $n > 2$ even in probability sense. If $m \in \mathbb{N}^*$ and $2 + \frac{1}{m+1} \leq \delta < 2 + \frac{1}{m}$, then the flow of $BES^x(\delta)$ processes has exactly $2 + m$ derivatives in P -a.s. sense. We remark that the regularity of the flow is increasing as $\delta \downarrow 2$, and for $\delta = 2$ the flow is C^∞ . The asymptotic relations a), b) give us logarithmic asymptotics for n -th derivative of ρ_t^x . The asymptotic relations c), d) characterize the behaviour of the ratio of the n -th and the first derivatives in P -a.s. and L^γ - sense.

If $1 < \delta < 2$, then $BES^x(\delta)$ process touches 0 with probability 1 and the results will be different from the previous case. To present the results let us denote as before for $x > 0$

$$\tau_0(x) = \inf\{s \geq 0 : \rho_s^x = 0\} \tag{5}$$

with $\inf\{\emptyset\} = +\infty$.

Theorem 1.3. *In the case $1 < \delta < 2$ and $x > 0$ there exists a modification $\tilde{\rho}$ of the Bessel flow in the space $D(\mathbb{R}^{+,*}, C(\mathbb{R}^{+,*}))$ with the following properties:*

- a) $\tilde{\rho}$ is bicontinuous P -a.s. and has bicontinuous derivatives of all orders on the set $]0, +\infty[\times]0, \tau_0(x)[$. These derivatives coincide with the ones of $(\rho_{t \wedge \tau_0(x)}^x)_{x>0, t>0}$.
- b) For each (x, t) with $x > 0, t > 0$ $\tilde{\rho}$ has derivatives in probability sense only up to the order $n < n(\delta)$ with $n(\delta) = \frac{1}{2-\delta}$, which are bicontinuous in probability. Moreover, for $n < n(\delta)$ we have:

$$P \lim_{y \rightarrow x+} \left(\ln \left(\frac{\partial^n \rho_{\tau_0(x)}^y}{\partial y^n} \right) / \ln \left(\rho_{\tau_0(x)}^y \right) \right) = n(\delta) - n,$$

- c) For $n < n(\delta)$ we have

$$P \lim_{y \rightarrow x+} \left((\rho_{\tau_0(x)}^y)^{n-1} \frac{\partial^n \rho_{\tau_0(x)}^y}{\partial y^n} / \frac{\partial \rho_{\tau_0(x)}^y}{\partial y} \right) = \frac{U_{n-1}}{x^{n-1}}$$

where U_{n-1} is the same as in Theorem 1.2 and U_1 is a random variable given by (24) with $\nu = 5 - 2\delta$.

- d) For $n < n(\delta)$ and $0 < \gamma < (5 - 2\delta)/(n - 1)$ with the same U_{n-1} and U_1 as in c)

$$\lim_{y \rightarrow x+} E \left((\rho_{\tau_0(x)}^y)^{n-1} \left| \frac{\partial^n \rho_{\tau_0(x)}^y}{\partial y^n} / \frac{\partial \rho_{\tau_0(x)}^y}{\partial y} \right| \right)^\gamma = \frac{E(|U_{n-1}|^\gamma)}{x^{\gamma(n-1)}}$$

In the case $1 < \delta < 2$ and $x \geq 0, t > 0$ the mentioned above modification of Bessel flow has the same regularity as for $x > 0, t > 0$.

Remark 2. For the regularity in probability sense for $x > 0$ and $t > 0$ we have the following picture. For $1 < \delta \leq 3/2$ the flow has only one derivative in probability sense. For $m \in \mathbb{N}^*$ and $2 - \frac{1}{m+1} < \delta \leq 2 - \frac{1}{m+2}$ the considered modification has exactly $m + 1$ derivatives in probability sense. We remark that the regularity in probability sense is increasing to infinity as $\delta \uparrow 2$. The interpretation of the asymptotic relations is the same as in Remark 1.

2 Regularity of Bessel flow for $\delta \geq 2$

In the case $x > 0$ we begin with some rather general Lemmas.

Lemma 2.1. *Let ρ^x be the strong unique (P -a.s.) solution of the equation (2) with initial value x . If for each $\gamma > 0$ the flow of ρ^x with $x > \gamma$ is bicontinuous and has (P -a.s.) bicontinuous in (x, t) derivatives of all orders with respect to x on the set $]\gamma, +\infty[\times]0, +\infty[$, then there exists an extension of the flow on the set $]0, +\infty[\times]0, +\infty[$ having the same properties.*

Proof A simple patching with respect to γ proves the result. \square

Let now $\gamma > 0$ be fixed and $x > \gamma > 0$. To localise the coefficients of the equation (2) we take a bicontinuous version of ρ^x (see [10], p.362). For $0 < \epsilon < \gamma$ we put

$$\tau_\epsilon = \inf\{t \geq 0 : \rho_t^\gamma \leq \epsilon\}, \quad (6)$$

and

$$\tau = \inf\{t \geq 0 : \rho_t^\gamma = 0\}, \quad (7)$$

with $\inf\{\emptyset\} = \infty$. To simplify the notations and since γ is fixed, we do not write that τ_ϵ and τ depend on γ .

Lemma 2.2. *Suppose for each $\epsilon > 0$ there exists a bicontinuous version of the flow of the process $(\rho_{t \wedge \tau_\epsilon}^x)_{t \geq 0}$ having (P-a.s.) bicontinuous in (x, t) derivatives with respect to x on the set $] \gamma, +\infty[\times] [0, \tau_\epsilon[$ then there exists an extension of the flow having the same properties on the set $] \gamma, +\infty[\times] [0, \tau[$.*

Proof A simple patching with respect to ϵ gives the result. \square

Proof of Theorem 1.1 Using the Lemmas 2.1, 2.2 and the fact that $P(\tau = \infty) = 1$, we reduce our study to the process $(\rho_{t \wedge \tau_\epsilon}^x)_{t \geq 0}$ with $x \in] \gamma, +\infty[$ and $\gamma > 0$, where τ_ϵ is defined by (6). By comparison theorem (P-a.s.) for all $t \geq 0$ and $x > \gamma$

$$\rho_{t \wedge \tau_\epsilon}^x \geq \rho_{t \wedge \tau_\epsilon}^\gamma \geq \epsilon.$$

For $x, y \in] \gamma, +\infty[$ we denote

$$Z_{t \wedge \tau_\epsilon}^{y,x} = \frac{\rho_{t \wedge \tau_\epsilon}^y - \rho_{t \wedge \tau_\epsilon}^x}{y - x}. \quad (8)$$

From (2) we obtain the following linear equation:

$$Z_{t \wedge \tau_\epsilon}^{y,x} = 1 - \frac{(\delta - 1)}{2} \int_0^{t \wedge \tau_\epsilon} \frac{Z_s^{y,x}}{\rho_s^y \rho_s^x} ds \quad (9)$$

and, hence, the solution

$$Z_{t \wedge \tau_\epsilon}^{y,x} = \exp \left\{ -\frac{(\delta - 1)}{2} \int_0^{t \wedge \tau_\epsilon} \frac{1}{\rho_s^y \rho_s^x} ds \right\}. \quad (10)$$

To take the limit as $y \rightarrow x$ we use bicontinuity of the flow of ρ^x , the fact that on the interval $] [0, t \wedge \tau_\epsilon[$ we have a minoration: $\rho_s^y \geq \epsilon$, $\rho_s^x \geq \epsilon$. By Lebesgue dominating convergence theorem, the first derivative of the flow is given by:

$$Y_{t \wedge \tau_\epsilon}^x = \exp \left\{ -\frac{(\delta - 1)}{2} \int_0^{t \wedge \tau_\epsilon} \frac{1}{(\rho_s^x)^2} ds \right\}. \quad (11)$$

The bicontinuity of the first derivative follows in the same way using bicontinuity of the flow ρ^x and the above minoration.

Now, using the expression of the first derivative and the arguments mentionned for bicontinuity of the first derivative, we prove, in recurrence way, the existence and bicontinuity of the n -th derivative. \square

To study the existence and bicontinuity of the derivatives of Bessel flow at $x = 0$, we need a reinforced scaling property and some asymptotic results.

Lemma 2.3. *If we consider bicontinuous versions of Bessel processes, then for all $c > 0$ the reinforced scaling property holds:*

$$\mathcal{L} \left(\frac{1}{c} (\rho_{c^2 t}^x)_{t \geq 0, x > 0} \right) = \mathcal{L} \left((\rho_t^{x/c})_{t \geq 0, x > 0} \right).$$

Proof This is clair result for a finite number of x , say x_1, x_2, \dots, x_n , due to simple scaling property of Bessel process and the uniqueness of the solution of (2). Then, the result follows by continuity. \square

Remark 3. As a corollary of this Lemma, the law of a measurable functional of $\left(\frac{1}{c} (\rho_{c^2 t}^x)_{t \geq 0, x > 0} \right)$ is the same as the law of the rescaled functional of $\left((\rho_t^{x/c})_{t \geq 0, x > 0} \right)$. In particular,

$$\mathcal{L} \left(\left(\int_0^t \frac{ds}{(\rho_s^x)^2} \right)_{t \geq 0, x > 0} \right) = \mathcal{L} \left(\left(\int_0^{t/x^2} \frac{ds}{(\rho_s^1)^2} \right)_{t \geq 0, x > 0} \right),$$

and we note using Cauchy sequence characterisation of P-a.s. convergence, that P-a.s. convergence for the rescaled processes is equivalent to the same for the original processes.

Lemma 2.4. *If $\delta = 2$ and $x \geq 0$, then as $t \rightarrow +\infty$,*

$$\frac{4}{(\ln t)^2} \int_1^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{\mathcal{L}} T_1(\beta)$$

where $T_1(\beta)$ is the first passage time of level 1 for a standard brownian motion.

If $\delta > 2$ and $x \geq 0$, then as $t \rightarrow +\infty$,

$$\frac{1}{\ln t} \int_1^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{a.s.} \frac{1}{(\delta - 2)}.$$

If $\delta = 2$ and $x \geq 0$, then as $t \rightarrow +\infty$,

$$\frac{1}{\ln t} \int_1^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{a.s.} +\infty.$$

Proof The first two asymptotics are well-known. For instance, we can find the proof of the first one in [8] and, the second and the third ones can be found in [3]. \square

Lemma 2.5. *If $\delta = 2$ and $t > 0$, then as $x \rightarrow 0+$,*

$$\frac{1}{(\ln x)^2} \int_0^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{\mathcal{L}} T_1(\beta)$$

where $T_1(\beta)$ is the first passage time of level 1 for a standard brownian motion.

If $\delta > 2$ and $t > 0$, then as $x \rightarrow 0+$,

$$\frac{1}{\ln x} \int_0^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{a.s.} \frac{2}{(2-\delta)}.$$

If $\delta = 2$ and $t > 0$, then as $x \rightarrow 0+$,

$$\frac{1}{\ln x} \int_0^t \frac{1}{(\rho_s^x)^2} ds \xrightarrow{a.s.} -\infty.$$

The mentionned a.s. convergences are uniform in t on compacts of $(0, +\infty)$.

Proof Take $x > 0$ and consider the integral on $[0, t]$. Make a change of variables $s = x^2 u$ and use scaling property of Lemma 2.3, Lemma 2.4 with $x = 1$ and the Remark 3. \square

Lemma 2.6. Let $\alpha \geq 0$ and $\beta \geq 0$. The integral

$$\int_0^{+\infty} \frac{(Y_s^1)^\alpha}{(\rho_s^1)^\beta} ds \tag{12}$$

is convergent P-a.s. iff $\alpha(\delta - 1) + (\beta - 2)(\delta - 2) > 0$. In particular, it is convergent when $\alpha + \beta > 2$. The mentionned convergence is uniform in t on compacts of $(0, +\infty)$.

Proof We notice that the integrand in (12) is positive and, hence, the integral is convergent P-a.s. to a finite limit or to $+\infty$. We remark that for $s \geq 0$ the first derivative of Bessel flow

$$Y_s^1 = \exp\left\{-\frac{(\delta - 1)}{2} \int_0^s \frac{1}{(\rho_u^1)^2} du\right\}.$$

By Ito formula we have:

$$\ln(\rho_s^1) = \int_0^s \frac{1}{\rho_u^1} d\beta_u + \frac{(\delta - 2)}{2} \int_0^s \frac{1}{(\rho_u^1)^2} du.$$

So, performing a time change with $A_s = \int_0^s \frac{1}{(\rho_u^1)^2} du$ in considered integral we obtain that:

$$\int_0^{+\infty} \frac{(Y_s^1)^\alpha}{(\rho_s^1)^\beta} ds \stackrel{\mathcal{L}}{=} \int_0^{+\infty} \exp\left(a\tilde{\beta}_u - \frac{b u}{2}\right) du$$

where $a = 2 - \beta$ and $b = \alpha(\delta - 1) + (\beta - 2)(\delta - 2)$ and $\tilde{\beta}$ standard Brownian motion. It remains to note that the last integral is convergent iff $b > 0$. \square

To prove the existence of the derivatives of Bessel flow at $x = 0$ we need an explicite expression for the derivatives of Bessel flow at $x > 0$. For this we introduce $h^x = (h_t^x)_{t \geq 0}$ with

$$h_t^x = -\frac{(\delta - 1)}{2} \int_0^t \frac{1}{(\rho_s^x)^2} ds. \tag{13}$$

Let now \mathcal{I}_n be a set of multi-indices:

$$\mathcal{I}_n = \{I = (i_1, i_2, \dots, i_n) : i_1 \geq 0, i_2 \geq 0, \dots, i_n \geq 0, \sum_{r=1}^n r i_r = n\}.$$

For $g \in C^n(\mathbb{R})$ and $I \in \mathcal{I}_n$, $I = (i_1, i_2, \dots, i_n)$, we introduce differential monomials

$$Q_I(g) = \left(\frac{\partial g}{\partial x}\right)^{i_1} \left(\frac{\partial^2 g}{\partial x^2}\right)^{i_2} \dots \left(\frac{\partial^n g}{\partial x^n}\right)^{i_n}. \quad (14)$$

as well as differential polynomials

$$P_n(g) = \sum_{I \in \mathcal{I}_n} c_I Q_I(g) \quad (15)$$

where c_I are real constants.

Then using the existence of the first derivative and recurrence arguments we obtain that for all $n \geq 1$

$$\frac{\partial^n \rho_{t \wedge \tau_\epsilon}^x}{\partial x^n} = Y_{t \wedge \tau_\epsilon}^x P_{n-1}(h_{t \wedge \tau_\epsilon}^x) \quad (16)$$

with $P_0(\cdot) = 1$ and τ_ϵ defined by (6). By the same reasoning we can prove that for all $1 \leq k < n$

$$\frac{\partial^k h_{t \wedge \tau_\epsilon}^x}{\partial x^k} = \sum_{I \in \mathcal{I}_k} c_I \int_0^{t \wedge \tau_\epsilon} \frac{Q_I(\rho_s^x)}{(\rho_s^x)^{2+j_k}} ds. \quad (17)$$

where $Q_I(\cdot)$ are the differential monomials of the type (14) and for $I = (i_1, i_2, \dots, i_k)$, $j_k = \sum_{r=1}^k i_r$. Taking the limit as $\epsilon \rightarrow 0$ and using the fact that $P(\tau_0 = \infty) = 1$ we obtain the needed formulas. These formulas coincide with the ones obtained by replacing of $t \wedge \tau_\epsilon$ by t in (16) and (17).

Now for $n \geq 1$ we introduce the integrals

$$b_n = x^n \int_0^\infty \frac{(Y_s^x)^n}{(\rho_s^x)^{n+2}} ds \quad (18)$$

which are convergent according to Lemma 2.6. Moreover, using change of variables $s = s'x^2$ we establish that

$$b_n \stackrel{\mathcal{L}}{=} \int_0^\infty \frac{(Y_s^1)^n}{(\rho_s^1)^{n+2}} ds$$

and that the law of b_n does not depend on x . Let $B_0 = 1$ and let us denote by B_n the quantity:

$$B_n = \sum_{k=1}^n b_k^{1/k} \quad (19)$$

We notice that the law of B_n does not depend on x since it is so for the b_k .

Lemma 2.7. *For each $n \geq 1$ there exists a real positive constant $c = c(n)$ such that*

$$|x^{n-1} \frac{\partial^n \rho_t^x}{\partial x^n}| \leq c Y_t^x B_{n-1}^{n-1}, \quad |x^n \frac{\partial^n h_t^x}{\partial x^n}| \leq c B_n^n.$$

As a consequence for each $n \geq 1$ there exists a constant $c = c(n)$ such that

$$|x^n P_n(h_t^x)| \leq c B_n^n.$$

Proof The proof is going by induction using previous formulas for the derivatives. For $n = 1$ we have

$$x \frac{\partial}{\partial x}(h_t^x) = (\delta - 1) x \int_0^t \frac{Y_s^x}{(\rho_s^x)^3} ds \quad (20)$$

and, hence,

$$|x \frac{\partial}{\partial x}(h_t^x)| \leq (\delta - 1) b_1.$$

So, we see that the claim is true with $c = \max(1, \delta - 1)$. Suppose that for $1 \leq m \leq n$

$$|x^{m-1} \frac{\partial^m \rho_t^x}{\partial x^m}| \leq c Y_t^x B_{m-1}^{m-1}, \quad |x^m \frac{\partial^m h_t^x}{\partial x^m}| \leq c B_m^m.$$

We show that the needed relations hold for $m = n + 1$. Below we will denote by c a generic constant. From the formula (16) with replacing n by $n + 1$ and (15) it follows that for the first estimation it is sufficient to majorate each $Q_I(h_t^x)$ with $I \in \mathcal{I}_n$ where

$$Q_I(h_t^x) = \left(\frac{\partial h_t^x}{\partial x}\right)^{i_1} \left(\frac{\partial^2 h_t^x}{\partial x^2}\right)^{i_2} \dots \left(\frac{\partial^n h_t^x}{\partial x^n}\right)^{i_n}$$

and $I = (i_1, i_2, \dots, i_n)$ with $\sum_{r=1}^n r i_r = n$. Since B_n is increasing sequence we obtain from our suppositions that

$$|x^n Q_I(h_t^x)| \leq c B_1^{i_1} B_2^{2i_2} \dots B_n^{ni_n} \leq c B_n^n$$

and it gives the first and third estimations of Lemma.

We remark that

$$\left| \frac{\partial^{n+1} h_t^x}{\partial x^{n+1}} \right| \leq c \sum_{I \in \mathcal{I}_{n+1}} \int_0^\infty \frac{|Q_I(\rho_s^x)|}{(\rho_s^x)^{j_{n+1}+2}} ds$$

with $I = (i_1, i_2, \dots, i_{n+1})$, $\sum_{r=1}^{n+1} r i_r = n + 1$ and $j_{n+1} = \sum_{r=1}^{n+1} i_r$.

Then we take in account the formula (16) to obtain

$$Q_I(\rho_s^x) = (Y_s^x)^{j_{n+1}} (P_1(h_s^x))^{i_2} \dots (P_n(h_s^x))^{i_{n+1}}.$$

Since $\sum_{r=2}^{n+1} (r-1) i_r = n + 1 - j_{n+1}$, we have from previous estimations that

$$|x^{n+1-j_{n+1}} Q_I(\rho_s^x)| \leq c (Y_s^x)^{j_{n+1}} B_1^{i_2} B_2^{2i_3} \dots B_n^{ni_{n+1}}.$$

Using this estimation and doing the change of variables in the integrals we obtain that

$$\left| x^{n+1} \frac{\partial^{n+1} h_t^x}{\partial x^{n+1}} \right| \leq c \sum_{I \in \mathcal{I}_{n+1}} b_{j_{n+1}} B_1^{i_2} B_2^{2i_3} \dots B_n^{ni_{n+1}} \leq c B_{n+1}^{n+1}$$

since $1 \leq j_{n+1} \leq n + 1$ and $b_r \leq B_r^r$ for each r . So, we have the second estimation and it proves Lemma. \square

Lemma 2.8. *Let $t > 0$ and $x > 0$. For all fixed $n \geq 1$ the sequences of random variables $(x^n \frac{\partial^n h_t^x}{\partial x^n})$ and $(x^n P_n(h_t^x))$ are convergent P -a.s. as $x \rightarrow 0+$. The mentioned convergences are uniform in t on compacts of $(0, +\infty)$.*

Proof We prove in recurrent way that for all $n \geq 1$ the sequence of random variables $(x^n \frac{\partial^n h_t^x}{\partial x^n})$ is convergent P-a.s. as $x \rightarrow 0+$ uniformly in t on compacts of $(0, +\infty)$. We remark that by formulas (14), (15) this convergence gives immediately the same type of convergence for $(x^n P_n(h_t^x))$.

For $n = 1$ we have

$$x \frac{\partial}{\partial x}(h_t^x) = (\delta - 1)x \int_0^t \frac{Y_s^x}{(\rho_s^x)^3} ds \stackrel{\mathcal{L}}{=} (\delta - 1) \int_0^{t/x^2} \frac{Y_s^1}{(\rho_s^1)^3} ds. \quad (21)$$

By Lemma 2.6 with $\alpha = 1$ and $\beta = 3$ we prove that the integral is convergent P-a.s., uniformly in t on compacts of $(0, +\infty)$.

Suppose that P-a.s. convergence, uniform in t on compacts of $(0, +\infty)$ is valid for $(x^k \frac{\partial^k h_t^x}{\partial x^k})$ with $1 \leq k \leq n$. Then by (15) we obtain that the $(x^k P_k(h_t^x))$ are convergent P-a.s., uniformly in t on compacts of $(0, +\infty)$, as $x \rightarrow 0+$. By formula (17) we have:

$$\frac{\partial^{n+1} h_t^x}{\partial x^{n+1}} = \sum_{I \in \mathcal{I}_{n+1}} c_I \int_0^t \frac{Q_I(\rho_s^x)}{(\rho_s^x)^{2+j_{n+1}}} ds.$$

where $I = (i_1, i_2, \dots, i_{n+1})$, $\sum_{r=1}^{n+1} r i_r = n+1$ and $j_{n+1} = \sum_{r=1}^{n+1} i_r$. We show that each term in the previous sum multiplying by x^{n+1} is convergent P-a.s. uniformly in t on compacts of $(0, \infty)$. For this we remark that the term corresponding to $I = (i_1, i_2, \dots, i_{n+1})$ in the previous sum times x^{n+1} is equal to

$$x^{n+1} \int_0^t \frac{(Y_s^x)^{j_{n+1}} (P_1(h_s^x))^{i_2} (P_2(h_s^x))^{i_3} \dots (P_n(h_s^x))^{i_{n+1}}}{(\rho_s^x)^{2+j_{n+1}}} ds$$

and that it is equal in law to

$$\int_0^{t/x^2} \frac{(Y_s^1)^{j_{n+1}} (P_1(h_{s/x^2}^1))^{i_2} (P_2(h_{s/x^2}^1))^{i_3} \dots (P_n(h_{s/x^2}^1))^{i_{n+1}}}{(\rho_s^1)^{2+j_{n+1}}} ds$$

where the $P_k(h_{s/x^2}^1)$ are equal in law to the $x^k P_k(h_s^x)$.

We notice that for $1 \leq k \leq n$ the $(x^k P_k(h_t^x))$ are uniformly bounded by B_k^k according to Lemma 2.7 and that the law of B_k^k does not depend on x . Moreover, since $j_{n+1} \geq 1$, the integral

$$\int_0^\infty \frac{(Y_s^x)^{j_{n+1}}}{(\rho_s^x)^{2+j_{n+1}}} ds$$

is converging. So, changing space, we have P-a.s. convergence by Lebesgue dominated convergence theorem. Then, the final result follows by Lemma 2.3.

The same can be done simultaneously for the expression of $(x^{n+1} \frac{\partial^{n+1} h_t^x}{\partial x^{n+1}})$ and this proves P-a.s. convergence of this variable, uniform in t on compacts of $(0, +\infty)$. \square

Lemma 2.9. For $n \geq 1$ and $t > 0$ let $U_n = \lim_{x \rightarrow 0+} x^n P_n(h_t^x)$. Then $U_n \neq 0$ (P-a.s.).

Proof Writing the expression for $(n+2)$ -th derivative of ρ_t^y from (16) and derivating the same expression for $(n+1)$ -th derivative of ρ_t^y we get that: for $y > 0$ and $t > 0$

$$P_{n+1}(h_t^y) = \frac{\partial}{\partial y}(h_t^y)P_n(h_t^y) + \frac{\partial}{\partial y}(P_n(h_t^y)).$$

We notice that for $n \geq 1$

$$\lim_{y \rightarrow 0+} y^{n+1}P_{n+1}(h_t^y) = U_{n+1}, \quad \lim_{y \rightarrow 0+} y^n P_n(h_t^y) = U_n, \quad \lim_{y \rightarrow 0+} y \frac{\partial}{\partial y}(h_t^y) = U_1$$

and we prove that

$$U_{n+1} = (U_1 - n)U_n. \quad (22)$$

For this we take $x > 0$ and $\alpha \in]0, 1[$ and we integrate the previous equality on the interval $[\alpha x, x]$:

$$\int_{\alpha x}^x P_{n+1}(h_t^y)dy = \int_{\alpha x}^x \frac{\partial}{\partial y}(h_t^y)P_n(h_t^y)dy + P_n(h_t^x) - P_n(h_t^{\alpha x}).$$

Then we estimate each integral, we multiply the result by x^n , we take the limit as $x \rightarrow 0+$ and we obtain (22).

Finally, we have

$$U_{n+1} = U_1(U_1 - 1)(U_1 - 2) \cdots (U_1 - n) \quad (23)$$

with

$$U_1 = (\delta - 1) \int_0^\infty \frac{Y_s^1}{(\rho_s^1)^3} ds.$$

We show that the random variable U_1 has a density. For this we make a change of variable as in Lemma 2.6 with $A_s = \int_0^s \frac{1}{(\rho_u^1)^2} du$ to prove that

$$U_1 \stackrel{\mathcal{L}}{=} (\delta - 1) \int_0^\infty \exp\left(\beta_u - \frac{(2\delta - 3)}{2}u\right) du$$

where β is Brownian motion. Using Dufresne equality (see for instance [15], p. 95) we have

$$U_1 \stackrel{\mathcal{L}}{=} \frac{2(\delta - 1)}{Z_\nu} \quad (24)$$

where Z_ν follows gamma law $\Gamma(\nu, 1)$ of index $\nu = (2\delta - 3)$.

Since U_1 has a density, $P(U_1 \in \mathbb{N}) = 0$ and for each $n \geq 1$, $U_n \neq 0$ (P-a.s.). \square

Proof of Theorems 1.2 We have from (2)

$$\rho_t^0 = \beta_t + \frac{(\delta - 1)}{2} \int_0^t \frac{1}{\rho_s^0} ds.$$

For $x > 0$ and $t > 0$ we put

$$Z_t^x = \frac{\rho_t^x - \rho_t^0}{x},$$

and we remark that Z_t^x satisfies a linear stochastic equation with the solution given by:

$$Z_t^x = \exp\left\{-\frac{(\delta - 1)}{2} \int_0^t \frac{1}{\rho_s^x \rho_s^0} ds\right\}.$$

The fact that for all $s \geq 0$, $\rho_s^x \downarrow \rho_s^0$ (P-a.s.) as $x \downarrow 0$ and the property: P-a.s. for all $t > 0$

$$\int_0^t \frac{1}{(\rho_s^0)^2} ds = +\infty, \quad (25)$$

together with Lebesgue monotone convergence theorem give that P-a.s. for all $t > 0$

$$\frac{\partial \rho_t^x}{\partial x} \Big|_{x=0} = \lim_{x \rightarrow 0+} Z_t^x = 0.$$

In the same manner we establish that P-a.s. for all $t_0 > 0$

$$\lim_{\substack{x \rightarrow 0+ \\ t \rightarrow t_0}} \frac{\partial \rho_t^x}{\partial x} = \lim_{\substack{x \rightarrow 0+ \\ t \rightarrow t_0}} \exp\left\{-\frac{(\delta-1)}{2} \int_0^t \frac{1}{(\rho_s^x)^2} ds\right\} = 0$$

and this proves the continuity of the first derivative.

To prove the existence of the n -th derivative at $x = 0$ equal to zero we show that there exists $n(\delta)$ such that for $2 \leq n < n(\delta)$ (P-a.s.)

$$\lim_{x \rightarrow 0+} \frac{1}{x} \frac{\partial^{n-1} \rho_t^x}{\partial x^{n-1}} = 0. \quad (26)$$

To find $n(\delta)$ we write that

$$\frac{\partial^{n-1} \rho_t^x}{\partial x^{n-1}} = Y_t^x P_{n-2}(h_t^x)$$

where h^x is defined by (13) and P_n is given by (15). From Lemma 2.8 we have that uniformly in t on compact sets of $(0, +\infty)$

$$\lim_{x \rightarrow 0+} x^{n-2} P_{n-2}(h_t^x) = U_{n-2}$$

where U_{n-2} is different from zero with probability 1. This means that (26) is equivalent to

$$\lim_{x \rightarrow 0+} \exp\left\{-\left[\frac{(\delta-1)}{2} \int_0^t \frac{1}{(\rho_s^x)^2} ds + (n-1) \ln x\right]\right\} = 0 \quad (27)$$

If $\delta = 2$ then applying Lemma 2.5 we see that the last relation holds for all $n \geq 2$ and we can put $n(\delta) = +\infty$. If $\delta > 2$ then it is easy to see that for $n(\delta) = 1 + \frac{\delta-1}{\delta-2}$ and $n < n(\delta)$ the relation (27) holds and for $n > n(\delta)$ it fails. If $\delta > 2$ and $n = n(\delta)$ then by Ito formula

$$\ln(\rho_t^x) = \ln x + \int_0^t \frac{1}{\rho_s^x} d\beta_s + \frac{(\delta-2)}{2} \int_0^t \frac{1}{(\rho_s^x)^2} ds.$$

We apply a central limit theorem (see [12], p.472) for the martingale for $M = (M_t)_{t \geq 0}$ with

$$M_t = \frac{1}{\sqrt{|\ln(x)|}} \int_0^{t/x^2} \frac{1}{\rho_s^1} d\beta_s$$

obtained from original one by time change, to prove via Skorohod representation theorem that the quantity appearing as the power in exponential in (27), namely

$$\frac{(\delta-1)}{2} \int_0^t \frac{1}{(\rho_s^x)^2} ds + (n(\delta)-1) \ln x$$

behaves as $c\xi|\ln x|^{1/2}$ as $x \rightarrow 0$ where ξ is standard $\mathcal{N}(0, 1)$ random variable and c is some positive constant. Hence, (27) fails on the set $\{\xi > 0\}$ of probability $1/2$, as well as (26).

For the bicontinuity of the n -th derivative at $x = 0$ for $2 \leq n < n(\delta)$ we prove that P-a.s. for all $t_0 > 0$

$$\lim_{\substack{x \rightarrow 0+ \\ t \rightarrow t_0}} \frac{\partial^n \rho_t^x}{\partial x^n} = 0. \quad (28)$$

The proof of (28) is going in the same way as (26) using the fact that the convergences in Lemmas 2.8 and 2.5 are uniform in t on compact sets of $(0, +\infty)$.

The asymptotic relations a), a') b), c) follows from (16) and Lemmas 2.5 and 2.9. To prove d) we remark that according to Lemma 2.7 we have

$$\sup_{\epsilon \leq t \leq T} |x^{n-1} P_{n-1}(h_t^x)| \leq c B_{n-1}^{n-1}$$

with B_n defined by (19). It remains only to show that $B_{n-1}^{\gamma(n-1)}$ is integrable. Since for $a, b \in \mathbb{R}^+$ and $\gamma > 0$, $(a + b)^\gamma \leq c(a^\gamma + b^\gamma)$ with some constant c , $B_{n-1}^{(n-1)\gamma}$ is integrable if $b_k^{\gamma(n-1)/k}$ are integrables for $1 \leq k \leq n-1$. Making time change with $A_t = k^2 \int_0^t \frac{1}{(\rho_s^1)^2} ds$ and using Dufresne identity (see [2], p.78) we obtain that

$$b_k \stackrel{\mathcal{L}}{=} \frac{1}{k^2} \int_0^\infty \exp(\beta_u - \frac{(2\delta-3)u}{2k}) du \stackrel{\mathcal{L}}{=} \frac{2}{k Z_{\nu(k)}}$$

where $(\beta_u)_{u \geq 0}$ is standard Brownian motion and $Z_{\nu(k)}$ is the variable following gamma law $\Gamma(\nu(k), 1)$ of index $\nu(k) = \frac{2\delta-3}{k}$. So, we have needed integrability if for all $1 \leq k \leq n-1$, the variables $(2/Z_{\nu(k)})^{\frac{\gamma(n-1)}{k}}$ are integrables. As well-known this is true, if $\frac{\gamma(n-1)}{k} < \nu(k)$ and the last condition is satisfied for $\gamma < \frac{2\delta-3}{n-1}$. \square

3 Regularity of Bessel flow for $1 < \delta < 2$

We start with some lemmas needed to prove the theorem 1.3. Let $\tau_0(x)$ be defined by (5).

Lemma 3.1. *Let $x > 0$ be fixed. Then $\tau_0(x-) = \tau_0(x) = \tau_0(x+)$ (P-a.s.). Moreover, there exists a cadlag version of $(\tau_0(x))_{x>0}$.*

Proof By comparison theorem we have that for $x \leq y$ (P-a.s.)

$$\tau_0(x) \leq \tau_0(y).$$

Then, there exist the limits (P-a.s.):

$$\lim_{y \rightarrow x+} \tau_0(x) = \tau_0(x+), \quad \lim_{y \rightarrow x-} \tau_0(x) = \tau_0(x-).$$

We take $y < x < z$ then by comparison theorem again for $\gamma > 0$

$$E(\tau_0(y)^\gamma) \leq E(\tau_0(x)^\gamma) \leq E(\tau_0(z)^\gamma)$$

Since

$$\tau_0(x) \stackrel{\mathcal{L}}{=} \frac{x^2}{2\gamma_\nu}, \quad (29)$$

where γ_ν is random variable of gamma law $\Gamma(\nu, 1)$ with index $\nu = 1 - \frac{\delta}{2}$, we have for small $\gamma > 0$ that

$$E(\tau_0(x))^\gamma = c_\gamma x^{2\gamma}$$

with some positive constant c_γ . The mentionned informations implies that for small $\gamma > 0$

$$E(\tau_0(x-))^\gamma = E(\tau_0(x)^\gamma) = E(\tau_0(x+))^\gamma$$

and, hence, P-a.s. $\tau_0(x-) = \tau_0(x) = \tau_0(x+)$.

To construct a cadlag version of $(\tau_0(x))_{x>0}$, we take for $x \in \mathbb{Q}^+$ the value of $\tau_0(x)$ and for $x \in \mathbb{R}^+ \setminus \mathbb{Q}^+$ we put:

$$\tau_0(x-) = \lim_{\substack{y \rightarrow x- \\ y \in \mathbb{Q}^+}} \tau_0(y), \quad \tau_0(x) = \lim_{\substack{y \rightarrow x+ \\ y \in \mathbb{Q}^+}} \tau_0(y).$$

This construction gives a cadlag version of $(\tau_0(x))_{x>0}$ which preserves the finite-dimensional distributions of $\tau_0(x)$. \square

Lemma 3.2. *We consider bicontinuous (P-a.s.) modifications of $(\rho_t^x)_{t \geq 0, x > 0}$. Then the flow of processes $(\rho_{t \wedge \tau_0(x)}^x)_{t \geq 0}$ is bicontinuous in probability and has the first derivative in probability sense with respect to x . This derivative is bicontinuous in probability on the set $]0, +\infty[\times]0, +\infty[$.*

Proof Let (x, t) be fixed with $x > 0, t \geq 0$. To prove bicontinuity in probability of $(\rho_{t \wedge \tau_0(x)}^x)_{x>0, t \geq 0}$, we write:

$$|\rho_{s \wedge \tau_0(y)}^y - \rho_{t \wedge \tau_0(x)}^x| \leq |\rho_{s \wedge \tau_0(y)}^y - \rho_{s \wedge \tau_0(y)}^x| + |\rho_{s \wedge \tau_0(y)}^x - \rho_{t \wedge \tau_0(x)}^x|.$$

We introduce the set $A_\delta = \{|\tau(x) - \tau(y)| \leq \delta\}$ with $0 < \delta < 1$. Then, on the set A_δ for $|s - t| < \gamma < \delta$ we have:

$$|\rho_{s \wedge \tau_0(y)}^y - \rho_{t \wedge \tau_0(x)}^x| \leq \sup_{0 \leq u \leq t+\delta} |\rho_u^y - \rho_u^x| + \sup_{|u-v| \leq \delta; u, v \leq t+\delta} |\rho_u^x - \rho_v^x|.$$

We write for $\epsilon > 0$ that

$$P(|\rho_{s \wedge \tau_0(y)}^y - \rho_{t \wedge \tau_0(x)}^x| \geq \epsilon) \leq P(\{|\rho_{s \wedge \tau_0(y)}^y - \rho_{t \wedge \tau_0(x)}^x| \geq \epsilon\} \cap A_\delta) + P(A_\delta^c).$$

From the previous estimations we obtain that

$$P(|\rho_{s \wedge \tau_0(y)}^y - \rho_{t \wedge \tau_0(x)}^x| \geq \epsilon) \leq P(\sup_{0 \leq u \leq t+1} |\rho_u^y - \rho_u^x| \geq \epsilon/2) + P(\sup_{|u-v| \leq \delta; u, v \leq t+1} |\rho_u^x - \rho_v^x| \geq \epsilon/2) + P(A_\delta^c).$$

Using the facts that $(\rho_u^x)_{u \geq 0, x > 0}$ is bicontinuous P-a.s. and that $\tau(x)$ is continuous in probability, we obtain taking $\lim_{\delta \rightarrow 0} \lim_{y \rightarrow x}$, the claimed bicontinuity.

We show that for $t \in]0, +\infty[$ the first derivative of $(\rho_{t \wedge \tau_0(x)}^x)_{t \geq 0, x > 0}$ with respect to x is given by:

$$Y_{t \wedge \tau_0(x)}^x = \exp\left\{-\frac{(\delta-1)}{2} \int_0^{t \wedge \tau_0(x)} \frac{ds}{(\rho_s^x)^2}\right\} \quad (30)$$

For this we take $\epsilon > 0$ and we do a localisation with

$$\tau_\epsilon(x) = \inf\{s \geq 0 : \rho_s^x \leq \epsilon\}.$$

Then we write (10). We notice that $\tau_0(x) = \lim_{\epsilon \rightarrow 0} \tau_\epsilon(x)$ and, then,

$$Z_{t \wedge \tau_0(x)}^{x,y} = \exp\left\{-\frac{(\delta-1)}{2} \int_0^{t \wedge \tau_0(x)} \frac{ds}{\rho_s^x \rho_s^y}\right\}. \quad (31)$$

where $Z_t^{x,y}$ is defined by (8). Via comparison theorem it can be shown that

$$\lim_{y \rightarrow x} \int_0^{t \wedge \tau_0(x)} \frac{ds}{\rho_s^x \rho_s^y} = \int_0^{t \wedge \tau_0(x)} \frac{ds}{(\rho_s^x)^2}$$

and taking $\lim_{y \rightarrow x}$ in (31) we have (30). To obtain the result for $t = \infty$ it is sufficient to take $\lim_{t \rightarrow \infty}$ in (10) and continue in above way.

By time reversal we have:

$$\mathcal{L}\left((\rho_{\tau_0(x)-t}^{x,\delta})_{0 \leq t \leq \tau_0(x)}\right) = \mathcal{L}\left((\rho_t^{0,4-\delta})_{0 \leq t \leq L(x)}\right) \quad (32)$$

where $\rho^{x,\delta}$ is $\text{BES}^x(\delta)$ process and $L(x) = \sup\{t \geq 0 : \rho_t^{0,4-\delta} = x\}$. Since $1 < \delta < 2$, we have that $4 - \delta > 2$ and using asymptotics of Lemma 2.5 we obtain: (P-a.s.)

$$\int_0^{\tau_0(x)} \frac{ds}{(\rho_s^x)^2} = +\infty \quad (33)$$

and it gives together with (30) the expression

$$Y_{t \wedge \tau_0(x)}^x = \begin{cases} \exp\left\{-\frac{(\delta-1)}{2} \int_0^{t \wedge \tau_0(x)} \frac{ds}{(\rho_s^x)^2}\right\} & \text{if } t < \tau_0(x), \\ 0 & \text{if } t \geq \tau_0(x). \end{cases}$$

Now we prove a bicontinuity of the first derivative at each point (x, t) with $x > 0, t > 0$. We consider tree sets D_1, D_2, D_3 :

$$D_1 = \{\omega : \tau_0(x) > t\}, \quad D_2 = \{\omega : \tau_0(x) < t\}, \quad D_3 = \{\omega : \tau_0(x) = t\} \quad (34)$$

and we prove a bicontinuity on each of them. For D_1 we write that $D_1 = \bigcup_{\epsilon > 0} D_1^\epsilon$ where $D_1^\epsilon = \{\tau_\epsilon(x) > t\}$. On each set D_1^ϵ bicontinuity of $Y_{t \wedge \tau_0(x)}^x$ follows from bicontinuity (P-a.s.) of $(\rho_t^x)_{t > 0, x > 0}$. Hence, taking a countable set of ϵ we obtain the same result on D_1 . On the set D_2 we have $Y_{t \wedge \tau_0(x)}^x = Y_{\tau_0(x)}^x = 0$, and, hence it is continuous.

Take now the set D_3 , then $t = \tau_0(x)$. Let (s, y) be in the neighbourhood of $(\tau_0(x), x)$. We show that $Y_{s \wedge \tau_0(y)}^y$ is in the neighbourhood of $Y_{\tau_0(x)}^x = 0$. In fact, on the set $\{s \geq \tau_0(y)\}$, $Y_{s \wedge \tau_0(y)}^y = Y_{\tau_0(y)}^y = 0$. On the set $\{s < \tau_0(y)\}$ we remark that for all $x > 0$ (P-a.s.)

$$\lim_{\substack{y \rightarrow x \\ s \rightarrow \tau_0(x)-}} \int_0^s \frac{du}{(\rho_u^y)^2} = +\infty \quad (35)$$

In fact, by comparison theorem for small $\gamma > 0$ and $\epsilon > 0$ we have

$$\lim_{\substack{y \rightarrow x \\ s \rightarrow \tau_0(x)-}} \int_0^s \frac{du}{(\rho_u^y)^2} \geq \lim_{y \rightarrow x} \int_0^{(\tau_0(x)-\gamma) \wedge \tau_\epsilon(x)} \frac{du}{(\rho_u^y)^2} = \int_0^{(\tau_0(x)-\gamma) \wedge \tau_\epsilon(x)} \frac{du}{(\rho_u^x)^2} \quad (36)$$

Taking $\epsilon \rightarrow 0$ and then $\gamma \rightarrow 0$ we have from (33) and (36) the relation (35). \square

To investigate the existence of the derivatives of higher order, we prove the following two lemmas.

Lemma 3.3. *Let $x > 0$ be fixed. Then*

$$P \lim_{y \rightarrow x} \frac{\tau_0(y) - \tau_0(x)}{(y - x)^2} = 0.$$

Proof Let $y > x > 0$. Since $(\rho_{\tau_0(x)}^z, z > 0)$ is $\mathcal{F}_{\tau_0(x)}$ - Markov, we have

$$\tau_0(y) - \tau_0(x) = \inf\{u > 0 : \rho_u^{\rho_{\tau_0(x)}^y}\} = \tau_0(\rho_{\tau_0(x)}^y) \quad (37)$$

Using (29) we obtain that

$$\mathcal{L}(\tau_0(\rho_{\tau_0(x)}^y)) = \mathcal{L}((\rho_{\tau_0(x)}^y)^2 \frac{1}{2\gamma_\nu}). \quad (38)$$

where γ_ν is gamma random variable with index $\nu = 1 - \delta/2$, independent from $\rho_{\tau_0(x)}^y$. But

$$\frac{(\rho_{\tau_0(x)}^y)^2}{(y - x)^2} = \frac{(\rho_{\tau_0(x)}^y - \rho_{\tau_0(x)}^x)^2}{(y - x)^2} \quad (39)$$

since $\rho_{\tau_0(x)}^x = 0$. It was shown in Lemma 3.2 that (P-a.s.)

$$\lim_{y \rightarrow x} \frac{\rho_{\tau_0(x)}^y - \rho_{\tau_0(x)}^x}{y - x} = 0.$$

Then, (37), (38) and (39) implies

$$P \lim_{y \rightarrow x+} \frac{\tau_0(y) - \tau_0(x)}{(y - x)^2} = 0.$$

The same consideration with $x > y > 0$ gives again (39) with the exchanging x and y . But

$$\frac{\rho_{\tau_0(y)}^x - \rho_{\tau_0(y)}^y}{x - y} = Z_{\tau_0(y)}^{x,y} = \exp\left\{-\frac{(\delta - 1)}{2} \int_0^{\tau_0(y)} \frac{ds}{\rho_s^x \rho_s^y}\right\},$$

and the relation ($P - a.s.$)

$$\lim_{y \rightarrow x-} \int_0^{\tau_0(y)} \frac{ds}{\rho_s^x \rho_s^y} \rightarrow +\infty$$

implies

$$P \lim_{y \rightarrow x-} \frac{\tau_0(y) - \tau_0(x)}{(y - x)^2} = 0$$

and it proves the result. \square

Lemma 3.4. *The flow of the processes $(Y_{t \wedge \tau_0(x)}^x)_{t>0}$ defined by (30) has bicontinuous derivatives in probability sense at $x > 0$ only up to the order $n < n(\delta)$ where $n(\delta) = \frac{\delta - 1}{2 - \delta}$.*

Proof First of all we remark that $D_1 = \bigcup_{\epsilon>0} D_1^\epsilon$ and on the sets D_1^ϵ the flow of $(Y_{t \wedge \tau_0(x)}^x)_{t>0}$ with $x > 0$ has the bicontinuous derivatives of all orders. This follows from the fact that on D_1^ϵ this process coincide with $(Y_{t \wedge \tau_\epsilon(x)}^x)_{t>0, x>0}$ and we can use the previous results for classical case. On the set D_2 the result is also trivially true.

Let $(x, t) \in D_3$ be fixed with $x > 0$ and $t = \tau_0(x)$. Since $Y_{\tau_0(y)}^y = 0$ we evidently have that

$$P \lim_{y \rightarrow x-} \frac{\partial^n Y_{t \wedge \tau_0(y)}^y}{\partial y^n} = 0.$$

If we show that there exists $n(\delta) > 0$ such that for $n < n(\delta)$

$$P \lim_{y \rightarrow x+} \frac{\partial^n Y_{t \wedge \tau_0(y)}^y}{\partial y^n} = 0, \quad (40)$$

then the mentioned relations and continuity of the derivatives on D_1 and D_2 will imply that the flow of the processes $(Y_{t \wedge \tau_0(x)}^x)_{t>0}$ has continuous derivatives of the order $n < n(\delta)$. We recall that for $y > x$ and $t = \tau_0(x)$

$$\frac{\partial^n Y_{t \wedge \tau_0(y)}^y}{\partial y^n} = Y_{\tau_0(x)}^y P_n(h_{\tau_0(x)}^y) \quad (41)$$

where h^y and P_n are defined by (13),(15).

Let $u = y/x$. Performing time change $s = s'x^2$ we obtain

$$Y_{\tau_0(x)}^y P_n(h_{\tau_0(x)}^y) \stackrel{\mathcal{L}}{=} Y_{\tau_0(1)}^u P_n(h_{\tau_0(1)}^u) / x^n. \quad (42)$$

First of all we investigate the behaviour of

$$Y_{\tau_0(1)}^u = \exp \left\{ -\frac{(\delta - 1)}{2} \int_0^{\tau_0(1)} \frac{1}{(\rho_s^u)^2} ds \right\}.$$

Using time reversal we obtain that

$$\mathcal{L} \left(\int_0^{\tau_0(1)} \frac{1}{(\rho_s^u)^2} ds \mid \rho_{\tau_0(1)}^u = v \right) = \mathcal{L} \left(\int_0^{L_u(v)} \frac{1}{(\rho_s^{v, 4-\delta})^2} ds \right) \quad (43)$$

where $L_u(v) = \sup\{s \geq 0 : \rho_s^{v, 4-\delta} = u\}$. By time change we obtain that

$$\int_0^{L_u(v)} \frac{1}{(\rho_s^{v, 4-\delta})^2} ds \stackrel{\mathcal{L}}{=} \int_0^{L_{u/v}(1)} \frac{1}{(\rho_s^{1, 4-\delta})^2} ds \quad (44)$$

where $L_{u/v}(1)$ is defined as previously with replacing of u by u/v and v by 1. Since as $a \rightarrow +\infty$

$$\frac{L_a(1)}{a^2} \xrightarrow{\mathcal{L}} \tau_0(1)$$

where $\tau_0(1)$ is the corresponding time of attending of zero, we have that

$$P \lim_{a \rightarrow +\infty} \frac{\ln L_a(1)}{2 \ln a} = 1$$

Then using Lemma 2.4 we obtain that

$$P \lim_{v \rightarrow 0+} \frac{1}{|\ln(v)|} \int_0^{L_{u/v}(1)} \frac{1}{(\rho_s^{1,4-\delta})^2} ds = \frac{2}{2-\delta}. \quad (45)$$

Since $\lim_{u \rightarrow 1+} \rho_{\tau_0(1)}^u = 0$, we obtain using standard arguments from (45), (43) and (44) that

$$P \lim_{u \rightarrow 1+} \frac{1}{|\ln(\rho_{\tau_0(1)}^u)|} \int_0^{\tau_0(1)} \frac{1}{(\rho_s^u)^2} ds = \frac{2}{2-\delta}. \quad (46)$$

If we apply a time reversal to $P_n(h_{\tau_0(1)}^u)$ then we obtain

$$\mathcal{L}(P_n(h_{\tau_0(1)}^u) | \rho_{\tau_0(1)}^u = v) = \mathcal{L}(P_n(h_{L_u(v)}^{v,4-\delta})).$$

From Lemma 2.8 and 2.9 we obtain that

$$P \lim_{v \rightarrow 0+} v^n P_n(h_{L_u(v)}^{v,4-\delta}) = U_n$$

where U_n is defined by formula (23) with U_1 given by (24) and $\nu = 5 - 2\delta$. So, by standard arguments we deduce that

$$P \lim_{u \rightarrow 1+} (\rho_{\tau_0(1)}^u)^n P_n(h_{\tau_0(1)}^u) = U_n \quad (47)$$

Since $U_n \neq 0$ with probability 1, we obtain finally that the relation (40) is equivalent to

$$\lim_{u \rightarrow 1+} \exp \left\{ - \left[\frac{(\delta-1)}{2} \int_0^{\tau_0(1)} \frac{1}{(\rho_s^u)^2} ds + n \ln(\rho_{\tau_0(1)}^u) \right] \right\} = 0. \quad (48)$$

Let $n(\delta) = (\delta-1)/(2-\delta)$. It is easy to see from (46) and (47) that if $n < n(\delta)$ then (48) holds and that if $n > n(\delta)$ then (48) fails. For $n = n(\delta)$ it can be shown using a central limit theorem for martingales and Skorohod representation theorem like in the proof of theorems 1.2 that the limit in (48) exists only on the set of probability 1/2, and, that (48) fails, too.

□

Proof of theorem 1.3 Let ρ^0 be continuous version of $\text{BES}^0(\delta)$ process starting from zero and $(\tau_0(x))_{x>0}$ be a cadlag version of the corresponding process. For all $x > 0$ and $t \geq 0$ we put:

$$\tilde{\rho}_t^x = \rho_{t \wedge \tau_0(x)}^x I_{[0, \tau_0(x)]} + (\rho_t^0 - \rho_{\tau_0(x)}^0) I_{[\tau_0(x), +\infty[}. \quad (49)$$

Using strong Markov property we prove that the both processes ρ and $\tilde{\rho}$ have the same finite-dimensional distributions.

We show that the trajectories of (49) are in $D(\mathbb{R}^{+,*}, C(\mathbb{R}^{+,*}))$. For this we remark that for each $x > 0$, the process $(\tilde{\rho}_t^x)_{t \geq 0}$ is continuous in t . Moreover, for $y > x$

$$|\tilde{\rho}_t^y - \tilde{\rho}_t^x| \leq 2 \sup_{u,v \in U(x,y)} |\rho_u^y - \rho_v^x| + \sup_{u,v \in U(x,y)} |\rho_u^0 - \rho_v^0|$$

where $U(x, y) = \{(u, v) : |u - v| \leq (\tau_0(y) - \tau_0(x)); u, v \leq t + (\tau_0(y) - \tau_0(x))\}$. Since for all $x > 0$,

$$\lim_{y \rightarrow x+} \tau_0(y) = \tau_0(x+) = \tau_0(x),$$

we see, that uniformly on compact sets of t , the right-hand side of the last inequality is tending to zero as $y \rightarrow x+$. Taking $y < x$ and using the fact that for all $x > 0$

$$\lim_{y \rightarrow x-} \tau_0(y) = \tau_0(x-),$$

we obtain the existence of left-hand limits uniformly on compact set of t .

We show that the first derivative of $\tilde{\rho}$ coincide with the one of $(\rho_{t \wedge \tau_0(x)}^x)_{t \geq 0, x > 0}$. In fact, consider tree sets D_1, D_2, D_3 defined by (34). On the set D_1 the process $(\tilde{\rho}_t^x)_{t \geq 0}$ coincide with $(\rho_{t \wedge \tau_0(x)}^x)_{t \geq 0, x > 0}$, and the existence of the first derivative was already discussed in Lemmas 3.2, 3.4. On D_2 the same process coincide with $(\rho_t^0)_{t \geq \tau_0(x), x > 0}$ according to comparison theorem and the first derivative is equal to zero. On the set D_3 we have $t = \tau_0(x)$ and

$$\frac{\tilde{\rho}_t^y - \tilde{\rho}_t^x}{y - x} = \begin{cases} \frac{\rho_{\tau_0(x)}^y - \rho_{\tau_0(x)}^x}{y - x} & \text{if } y > x, \\ 0 & \text{if } y < x. \end{cases}$$

We obtain as in Lemma 3.3 that

$$\lim_{y \rightarrow x+} \frac{\rho_{\tau_0(x)}^y - \rho_{\tau_0(x)}^x}{y - x} = 0.$$

Hence, the first derivative of $\tilde{\rho}$ coincide with the one of $(\rho_{t \wedge \tau_0(x)}^x)_{t \geq 0, x > 0}$ and we obtain the claims from Lemma 3.4, and from the relations (41), (42), (46) and (47).

Now we consider the case $x = 0$ and $t > 0$. We put $(\tilde{\rho}_t^0)_{t \geq 0} = (\rho_t^0)_{t \geq 0}$ and we remark that for $x > 0$

$$\tilde{\rho}_t^x - \tilde{\rho}_t^0 = \rho_{t \wedge \tau_0(x)}^x - \rho_{t \wedge \tau_0(x)}^0 \quad (50)$$

Making time change and using scaling Lemma we have:

$$\frac{1}{x}(\rho_{t \wedge \tau_0(x)}^x - \rho_{t \wedge \tau_0(x)}^0) \stackrel{\mathcal{L}}{=} \rho_{(t/x^2) \wedge \tau_0(1)}^1 - \rho_{(t/x^2) \wedge \tau_0(1)}^0$$

Since $P(\tau_0(1) < \infty) = 1$, we obtain that for $t > 0$

$$P \lim_{x \rightarrow 0+} \frac{\tilde{\rho}_t^x - \tilde{\rho}_t^0}{x} = 0.$$

Then we verify easily a bicontinuity of the first derivative at $x = 0$ and $t > 0$. For $x > 0$ we have

$$Y_t^x = Y_{t \wedge \tau_0(x)}^x \stackrel{\mathcal{L}}{=} Y_{t/x^2 \wedge \tau_0(1)}^1.$$

Since $P(\tau_0(1) < \infty) = 1$ and (33) we obtain that for each $\epsilon > 0$

$$P(Y_{t/x^2 \wedge \tau_0(1)}^1 > \epsilon) = P(\tau_0(1) > t/x^2) \rightarrow 0$$

as $x \rightarrow 0+$.

From the formula (16) we have for $x > 0$ and $1 \leq n < n(\delta)$ that

$$\frac{\partial^n \tilde{\rho}_t^x}{\partial x^n} = \frac{\partial^{n-1} Y_{t \wedge \tau_0(x)}^x}{\partial x^{n-1}} = Y_{t \wedge \tau_0(x)}^x P_{n-1}(h_{t \wedge \tau_0(x)}^x)$$

and, hence,

$$\frac{\partial^n \tilde{\rho}_t^x}{\partial x^n} \stackrel{\mathcal{L}}{=} Y_{(t/x^2) \wedge \tau_0(1)}^1 P_{n-1}(h_{(t/x^2) \wedge \tau_0(1)}^1) / x^{n-1}.$$

We can see from the Lemma 2.6 and the proof of Lemma 2.8 that for $\delta > 3/2$, $P_{n-1}(h_{\tau_0(1)}^1)$ is a finite random variable. So, for each $\epsilon > 0$

$$P\left(\frac{1}{x} \frac{\partial^n \tilde{\rho}_t^x}{\partial x^n} > \epsilon\right) = P(\tau_0(1) > t/x^2) \rightarrow 0$$

as $x \rightarrow 0+$. It means that for $3/2 < \delta < 2$ there exist the bicontinuous derivatives of order $1 \leq n < n(\delta)$ with respect to x in probability sense at $x \geq 0$ and $t > 0$. \square

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